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FRACTIONAL VECTOR-ORDER h-REALISATION OF THE IMPULSE RESPONSE FUNCTION

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Abstract: The problem of realisation of linear control systems with the $\boldsymbol{h}$ −difference of Caputo-, Riemann–Liouville- and Grünwald–Letnikov-type fractional vector-order operators is studied. The problem of existing minimal realisation is discussed.

Key words: Realisation, Linear system, Fractional vector-order system, $\boldsymbol{h}$ −Markov parameters

1. INTRODUCTION

In engineering experiments, in many cases, only information about inputs and measurements of the investigated process are available. So, a relationship between these variables is needed. It is a question of the possible systems that provide a good description of observed system's input–output behaviour. This leads to the crucial idea of the realisation problem. In fact, the realisation of an input–output map describing a system's behaviour means finding a dynamical state-space system with input and output, which can be reproduced, when initialised at some state for the given (input–output) behaviour. In Bartosiewicz and Pawluszewicz (2006), classical conditions for existing realisation for continuous time or/and discrete time linear systems were generalised to any time domain. The next natural question is whether this problem can be extended to a more general case of differential/difference order, i.e. on systems defined by fractional order operators. The term fractional basically implies all non-integer numbers. In fact, in nature, there are many processes that can be more accurately modelled using fractional differ-integrals (see, e.g. in Ambroziak et al., 2016; Das, 2008; Koszewnik et al., 2016; Sierociuk et al., 2013; Wu et al., 2015). The rapid development of computer techniques has caused the parallel investigations in the field, among others, combinatorics tools and difference equations. This is the reason that in modelling of real phenomena, a generalisation of nth order differences to their fractional forms and the state-space description of control systems in discrete time are used (see, e.g. Bastos et al., 2011; Oprzedkiewicz and Gawin, 2016; Podlubny 1999).

The goal of this study is to construct a state-space fractional vector-order representation of an abstract input–output map and to give conditions under which such representation exists. To achieve this aim, fractional order h −differences of Caputo-, Riemann–Liouville- and Grünwald–Letnikov-type operators are considered. Taking into account their properties (Mozyrska et al., 2013), the state-space description of the system's behaviours is presented in terms of these operators parallel. The main result may be seen as an extension of the classical realisability criterion saying that an abstract input–output map has a state-space realisation if and only if the Markov parameters satisfy a recurrence relation (see Sontag, 1998; Zabczyk, 2008). To achieve this aim, h −Markov parameters for the input–output map are defined. It is shown that the input–output map has a state-space fractional vector-order h −realisation in finite number of steps if and only if the h −Markov parameters satisfy the linear recursion equation. The obtained relation is similar to the one given in the classical case; it extends the classical result to the fractional case. Generally, the obtained realisation is not unique; but under certain minimality or redundancy requirements, it can be what is a desirable property in practice. In fractional order case, some aspects of realisation problem were raised in Bettayeb et al. (2008), where the concept of the structured realisation index was introduced.

The paper is organised as follows. After introducing Mittag-Leffler function (Section 2.1) and fractional order difference operators (Section 2.2), the extension of controllability and observability conditions for fractional vector-order systems is presented (Section 3). In the next step, conditions of existing state-space fractional vector-order realisation are considered. As the last step, the problem of existing minimal fractional vector-order realisation is discussed (Section 4).

2. PRELIMINARIES

2.1. Discrete Mittag-Leffler function

Let α be any number and s any integer. Then:

$$\binom{\alpha}{s} = \begin{cases} 0 & for\ s < 0 \\ 1 & for\ s = 0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-s+1)}{s!} & for\ s > 0 \end{cases}$$

denotes the classical binomial coefficient. Denote the family of binomial functions by φ_μ parametrised by $\mu > 0$ as:

$$\varphi_\mu(n)=\begin{cases}\binom{n+\mu-1}{n} & for\ n\in N_0\\ 0 & for\ n<0.\end{cases} \tag{1}$$

If "$*$" denotes a convolution operator, then $(\varphi_\mu * \bar{x})(n) := \sum_{s=0}^{n}\binom{n-s+\mu-1}{n-s}\bar{x}(s)$, where $\bar{x}(s) := x(a+sh)$.

The discrete two-parameter Mittag-Leffler function is defined as (Mozyrska and Wyrwas, 2015; Mozyrska et al., 2017):

$$E_{(\alpha,\beta)}(\lambda,n) := \sum_{k=0}^{\infty}\lambda^k\varphi_{k\alpha+\beta}(n-k). \tag{2}$$

If A is $n\times n$ dimensional matrix with constant coefficients, then $E_{(\alpha,\beta)}(A,n) := \sum_{k=0}^{\infty}A^k\binom{n-k+k\alpha+\beta-1}{n-k}$ and $\varphi_{k\alpha+\beta}(n-k)=0$ for $n<k$. If $\alpha=\beta$, then $E_{(\alpha,\alpha)}(A,n)=\sum_{k=0}^{\infty}A^k\binom{n-(k+1)(\alpha-1)}{n-k}$. It is easy to check that even if there are matrices B and C such that $A=BC$, then $E_{(\alpha,\beta)}(BC,n)\neq E_{(\alpha,\beta)}(B,n)E_{(\alpha,\beta)}(C,n)$.

Let $A\in R^{n\times n}$ be a diagonalisable matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_{m_r}$ of multiples $m_1,m_2,\ldots,m_r$, respectively, such that $\sum_{i=1}^{r} m_i\le n$. Suppose that the function $f(\lambda)$ is well defined on the spectrum of the matrix A. Then a function of the matrix A is given by the Lagrange–Sylvester's interpolation formula (Gantmacher, 1959; Kaczorek, 1998) as:

$$f(A)=\sum_{i=1}^{r}[Z_{i1}\frac{df(\lambda)}{d\lambda}|_{\lambda=\lambda_i}+\cdots+Z_{i1}\frac{df^{(m_i-1)}(\lambda)}{d\lambda^{(m_i-1)}}|_{\lambda=\lambda_i}] \tag{3}$$

with coefficients:

$$Z_{ij}=\sum_{k=j-1}^{m_i-1}\frac{(A-\lambda_iI_n)^k\Phi(A)}{(k-j+1)!(j-1)!}\frac{df^{k-j+1}}{d\lambda^{k-j+1}}\left[\frac{1}{\Phi(\lambda)}\right]|_{\lambda=\lambda_i} \tag{4}$$

for $j=1,\ldots,m_r$, $\Phi(A)=\prod_{j=1}^{m_r}(A-\lambda_jI_n)^{m_i}$, $\Phi(\lambda)=\prod_{j=1}^{m_r}(\lambda-\lambda_j)^{m_i}$ and I_n – identity matrix of dimension $n\times n$.

Theorem 1 (Kaczorek, 2017): Let $\Phi(\lambda)=\det[\lambda I_n-f(A)]=\lambda^n+a_{n-a}\lambda^{n-a}+\cdots+a_1\lambda+a_0$, where $f(A)$ is given by (3), be the characteristic polynomial of matrix A. Then $f(A)$ satisfies its characteristic equation $[f(A)]^n+a_{n-1}[f(A)]^{n-1}+\cdots+a_1f(A)+a_0I_n=0$.

Proposition 2: Let $\Phi(\lambda)=\det[\lambda I_n-E_{(\alpha,\beta)}(A,k)]=\lambda^n+a_{n-a}\lambda^{n-a}+\cdots+a_1\lambda+a_0$ be the characteristic equation of the Mittag-Leffler function (2). Then matrix $\Phi(k)=E_{(\alpha,\beta)}(A,k)$ satisfies its characteristic equation $[E_{(\alpha,\beta)}(A,k)]^n+a_{n-1}[fE_{(\alpha,\beta)}(A,k)]^{n-1}+\cdots+a_0I_n=0$.

Proof: The reasoning using the Lagrange–Sylvester's formula (3), based on Kaczorek (2017), is the same as for one-parameter function Mittag-Leffler given in Pawluszewicz and Koszewnik (2019). □

From Proposition 2, immediately it follows that $E_{(\alpha,\beta)}(A,n) := \sum_{k=0}^{n}A^k\binom{n-k+k\alpha+\beta-1}{n-k}$.

2.2. Fractional h −difference operators

Let h be a positive real number. For any real a, let $(hN)_a=\{a,a+h,a+2h,\ldots\}$. Consider a function $x:(hN)_a\to R$. The forward h-difference operator is classically defined as $(\Delta_h x)(t)=\frac{x(t+h)-x(t)}{h}$. The n-fold application n of operator Δ_h, i.e. $\Delta_h^n:=\Delta_h\circ\ldots\circ\Delta_h$, for any natural n, leads to $(\Delta_h^n x)(t)=h^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x(t+kh)$. Additionally, we have $(\Delta_h^0 x)(t):=x(t)$. The fractional h −sum of order $\alpha>0$ for a function $x:(hN)_a\to R$ is defined by:

$$\left({}_a\Delta_h^{-\alpha}x\right)(t) := h^\alpha(\varphi_\alpha * \bar{x})(n),$$

where $t=a+(\alpha+n)h$ for any natural n.

Let $\alpha\in(0,1]$. The *Caputo-type* h *−difference operator* ${}_a\Delta_{h,*}^\alpha$ of order α for a function $x:(hN)_a\to R$ is defined as (Mozyrska and Girejko, 2013):

$$({}_a\Delta_{h,*}^\alpha x)(t) := ({}_a\Delta_h^{-(1-\alpha)}(\Delta_h x))(t) \tag{5}$$

for any $t\in(hN)_{a+(1-\alpha)h}$. If $\alpha=1$, then $({}_a\Delta_{h,*}^{\alpha=1}x)(t)=(\Delta_h x)(t)$ for any $t\in(hN)_a$. Note that $({}_a\Delta_{h,*}^\alpha x)(t)=h^{-\alpha}(\varphi_{1-\alpha} * \Delta_{h=1}\bar{x})(n)$ for any $t=a+(1-\alpha)h+nh$ and $\bar{x}(n)=x(a+nh)$.

The *Riemann–Liouville-type fractional* h *−difference operator* ${}_a\Delta_h^\alpha$ of order $\alpha\in(0,1]$ for a function $x:(hN)_a\to R$ is defined as (Bastos et al., 2011; Fereira and Torres, 2011):

$$\left({}_a\Delta_h^\alpha x\right)(t) := (\Delta_h\left({}_a\Delta_h^{-(1-\alpha)}x\right))(t),$$

where $t\in(hN)_{a+(1-\alpha)h}$.

The last operator we are considering is the *Grünwald–Letnikov-type fractional* h *−difference operator* ${}_a\tilde{\Delta}_h^\alpha$ of a real order α, defined for a function $x:(hN)_a\to R$ as (Mozyrska et al., 2013):

$$({}_a\tilde{\Delta}_h^\alpha x)(t) := \sum_{s=0}^{\frac{t-a}{h}}a_s^{(\alpha)}x(t-sh),$$

where $a_s^{(\alpha)}=(-1)^s\binom{\alpha}{s}\frac{1}{h^\alpha}$. If $a=(\alpha-1)h$. Then

$$({}_0\tilde{\Delta}_h^\alpha y)(t+h)=\left({}_a\Delta_h^\alpha x\right)(t), \tag{6}$$

where $x(t)=y(t-a)$ for $t\in(hN)_a$ (Mozyrska et al., 2013). Also, in Mozyrska et al. (2013), it was shown that for $\alpha\in(0,1]$,

$$({}_a\Delta_{h,*}^\alpha x)(t)=\left({}_a\Delta_h^\alpha x\right)(t)-\frac{x(a)}{h^\alpha}\binom{\frac{t-a}{h}}{-\alpha} \tag{7}$$

for $t\in(hN)_{a+(1-\alpha)h}$. Taking into account relations (6) and (7), one can use the common symbol defined by its values:

$$({}_a\Upsilon_h^\alpha x)(t)=\begin{cases}({}_a\Delta_{h,*}^\alpha x)(t)\ or\ \left({}_a\Delta_h^\alpha x\right)(t) & for\ a=(\alpha-1)h\\ ({}_a\tilde{\Delta}_h^\alpha x)(t+h) & for\ a=0.\end{cases}$$

Recall that the single-sided Z −transform of a sequence $\{y(n)\}_{n\in N_0}$ is a complex function $Y(z)$ given by $Y(z):=Z[y](z)=\sum_{k=0}^{\infty}\frac{y(k)}{z^k}$, where z is a complex variable for which series $\sum_{k=0}^{\infty}\frac{y(k)}{z^k}$ converges absolutely.

Proposition 3 (Mozyrska and Wyrwas, 2015): Let $a\in R$ and $\alpha\in(0,1]$. Define $y(n) := ({}_a\Upsilon_h^\alpha x)(t)$, where $t\in(hN)_{a+(1-\alpha)h}$ and $t=a+(1-\alpha)h+nh$. Then:

$$Z[({}_a\Upsilon_h^\alpha x)(t)](z)=z\left(\frac{hz}{z-1}\right)^{-\alpha}(X(z)-x(a)), \tag{8}$$

where $X(z)=Z[\bar{x}](z)$, $\bar{x}(n):=x(a+nh)$ and $\beta=\alpha$ for the Riemann–Liouville- or Grünwald–Letnikov-type h −difference operators and $\beta=1$ for the Caputo-type h −difference operator, and $a=\alpha-1$ for the Riemann–Liouville- or Caputo-type operators and $a=0$ for the Grünwald–Letnikov-type operator.

Proposition 4 (Mozyrska and Wyrwas, 2015): Let $\alpha \in (0,1]$. Then $Z\left[E_{(\alpha,\beta)}(\lambda,\cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta}\left(1-\frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1}$, where $|z|>1$ and $|z-1|^{\alpha}|z|^{1-\alpha} > |\lambda|$. Additionally $\beta=\alpha$ for the Riemann–Liouville- or Grünwald–Letnikov-type h −difference operators and $\beta = 1$ for the Caputo-type h −difference operator.

3. LINEAR FRACTIONAL VECTOR-ORDER SYSTEMS

Let us consider the following common form of vector-order $\alpha = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \in (0,1]$, $i = 1, \dots, p$, linear control systems initialised at time $t_0 \in (hN)_{t_0}$:

$$({}_{t_0}\Upsilon_h^{\alpha} x)(t) = Ax(t+t_0) + Bu(t) \tag{9a}$$

$$y(t) = Cx(t+t_0) \quad , \tag{9b}$$

where $x: (hN)_{t_0} \to R^p$ denotes a state vector, $y: (hN)_0 \to R^r$ an output vector, $u: (hN)_0 \to R^m$ a control, and $A \in R^{p\times p}$, $B \in R^{p\times m}$ and $C \in R^{r\times p}$ are real stationary matrices. Equation (9a) defines the dynamics of system (9) and Equation (9b) its output. Since matrices A, B, C for given $\alpha = (\alpha_1, \dots, \alpha_p)$ completly determinate system (9), shortly we will say that this system is described by the triple (A, B, C) for a given α. From definitions of fractional h-difference operators $({}_a\Delta_{h,*}^{\alpha} x)(t)$, $({}_a\Delta_h^{\alpha} x)(t)$ and $({}_a\widetilde{\Delta}_h^{\alpha} x)(t+h)$, it follows that dynamics (9a) can be rewritten as:

$$({}_{t_i}\Upsilon_h^{\alpha_i} x)(t_i) = h^{\alpha_i}\sum_{j=1}^{p} A_{ij}x_j\left(t_i + t_{0_i}\right) + h^{\alpha_i}\sum_{\iota=1}^{m} B_{\iota}u\_\iota(t_i)$$

for $i = 1, \dots, p$. Let $g^{\rho}(t) := g(t-h)$ for any $t \in (hN)_{t_0}$. For matrix $A \in R^{p\times p}$, define

$$E_{(\alpha,\beta)}(A,p) \coloneqq diag\{E_{(\alpha_i,\beta_i)}(A,p): i = 1, \dots, p\}. \tag{10}$$

Example 5: Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $p = 2$ and $\alpha = (\alpha_1, \alpha_2)$, such that $\alpha_1 = \beta_1 = 0{,}5$ and $\beta_2 = 0{,}5\alpha_2 = 0{,}5$. Then, for a positive h, we have:

$$E_{(\alpha,\beta)}(A,2) = diag\left\{E_{(\alpha_1,\beta_1)}(A,2), E_{(\alpha_2,\beta_2)}(A,2)\right\} =$$

$$diag\left\{\begin{pmatrix} \frac{3}{8} + (h^{0,5}-1)h^{0,5} & (1-2h^{0,5})h^{0,5} \\ 0 & \frac{3}{8} + (h^{0,5}-1)h^{0,5} \end{pmatrix}, \begin{pmatrix} \frac{3}{8} + (h^{0,25}-1)h^{0,25} & \left(\frac{3}{8} - 2h^{0,25}\right)h^{0,25}\Big) \\ 0 & \frac{3}{8} + (h^{0,25} - \frac{3}{4})h^{0,25} \end{pmatrix}\right\}.$$

Lemma 6: Let $\alpha = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \in (0,1]$, $i = 1, \dots, p$ and $\bar{A} = diag\{h^{-\alpha_i}A: i = 1, \dots, p\}$, $\bar{B} = \left(H^{-1}B \quad 0_{p\times m} \quad \cdots \quad 0_{p\times m}\right)^T$, $H := diag(h^{-\alpha_i}: i = 1, \dots, p)\}$. Dynamics of system (9) together with initial state $x_0 = \left(x(t_{0_1}) \quad \dots \quad x(t_{0_p})\right)^T = (x_{0_1} \quad \cdots \quad x_{0_p})^T = x_0 \in R^p$, $t_{0_i} = (\alpha_i - 1)h$, $i = 1, \dots, p$, and fixed controls u_ι, $\iota = 1, \dots, m$ has the unique solution:

$$x(t) = E_{(\alpha,\beta)}\left(\bar{A}, \frac{t-t_0}{h}\right)x_0 + \left(E_{(\alpha,\alpha)}^{\rho}(\bar{A},\cdot) * \bar{B}\bar{u}\right)\left(\frac{t-t_0}{h}\right),$$

where $\bar{u}\left(\frac{t-t_0}{h}\right) = h^{\alpha}u(t)$, and $\beta = 1$ for the fractional h −difference of Caputo-type operator, $\beta = \alpha$ for the fractional h −differences of Riemman–Liouville- and Grünwald–Letnikov-type operators.

Proof: Taking the Z −transform of both sides of Equation (9a), from Proposition 3 it follows that:

$$zh^{-\alpha_i}\left(1-\frac{1}{z}\right)^{\alpha_i}\left(X_i(z) - \left(\frac{z}{z-1}\right)^{\beta_i} x_i(t_{0_i})\right) = \sum_{j=1}^{p} A_{ij}X_j(z) + \sum_{\iota=1}^{m} B_\iota U_\iota(z), \tag{11}$$

where $Z\left[\bar{x}_i\left(\frac{t_i - t_{0_i}}{h}\right)\right](z) = X_i(z)$, $U_\iota\left[u_\iota\left(\frac{t_i - t_{0_i}}{h}\right)\right](z) = U\iota(z)$ for $i = 1, \dots, p$ and $\iota = 1, \dots, m$. Denoting $X(z) = (X_1(z) \quad \dots \quad X_p(z))^T$, $\Lambda_\alpha = diag\left\{\left(\frac{z}{z-1}\right)^{\alpha_i}: i = 1, \dots, p\right\}$ and $\Lambda_\beta = diag\left\{\left(\frac{z}{z-1}\right)^{\beta_i}: i = 1, \dots, p\right\}$, equation (11) can be rewritten as:

$$X(z) = \left(I_n - \frac{1}{z}\Lambda_\alpha HA\right)^{-1}\Lambda_\beta x_0 + \frac{1}{z}\left(I_n - \frac{1}{z}\Lambda_\alpha HA\right)^{-1}\Lambda_\alpha HBU(z).$$

If we put $F_1(z) = \left(I_n - \frac{1}{z}\Lambda_\alpha HA\right)^{-1}\Lambda_\beta$ and $F_2(z) = \left(I_n - \frac{1}{z}\Lambda_\alpha HA\right)^{-1}\Lambda_\alpha$, then $X(z) = F_1(z)x_0 + F_2(z)HB$. So, $\bar{x}(n) = Z^{-1}[X(z)]\left(\frac{t-t_0}{h}\right) = Z^{-1}[F_1(z)]\left(\frac{t-t_0}{h}\right)x_0 + Z^{-1}[F_2(z)U(z)]\left(\frac{t-t_0}{h}\right)$.

Since H, Λ_α and Λ_β are diagonal matrices, then by Proposition 4, one has $x_i\left(t_i + t_{0_i}\right) = E_{(\alpha_i,\beta_i)}\left(h^{-\alpha_i}A, \frac{t-t_0}{h}\right)x_0 + \left(E_{(\alpha_i,\alpha_i)}^{\rho}(\overline{h^{-\alpha_\iota}A},\cdot) * h^{-\alpha_i}B\bar{u}\right)\left(\frac{t-t_0}{h}\right)$. Taking into account (10), one obtains thesis. □

By $J_0(m)$, let us denote the set of all sequences $U = (u_0, u_1, \dots)$, where $u_n := u(t) = u(nh + t_0) \in \Omega$, $t \in (hN)_{t_0}$. Then, $\gamma(t+t_0, x_0, U) := x(t+t_0)$ will denote the state forward trajectory of system (9), i.e. a solution which is uniquely defined by the initial state x_0 and the control sequence $U \in J_0(m)$. The reachable set from the given initial state x_0 in q steps, denoted as $R_q(x_0)$, is the set of all states to which the given system can be steered from x_0 in q steps by the control sequence $U \in J_0(m)$, i.e. $R_q(x_0) := \{x \in R^p: x = \gamma(q, x_0, U), U \in J_0(m)\}$ with $R_0(x_0) := \{x_0\}$. Then, the set $R(x_0) := \cup_{q\in N_0} R_q(x_0)$ is the set of all states reachable from x_0.

Definition 7: System (9) is *locally controllable in q steps* from x_0 if there exists a neighbourhood $V \subset R^n$ of x_0, such that $V \subset R_q(x_0)$. System (9) is *globally controllable from x_0 in q steps* if $R_q(x_0) = R^p$.

Proposition 8: Let $\alpha = (\alpha_1, \dots, \alpha_p)$ with $\alpha_i \in (0,1]$, $i = 1, \dots, p$. Then system (9) is controllable in q steps if and only if the rank of controllability matrix $Q_q = \begin{pmatrix} \bar{\boldsymbol{B}} & E_{(\alpha,\alpha)}(\bar{A},1)\bar{B} & \dots & E_{(\alpha,\alpha)}(\bar{A},q-1)\bar{B} \end{pmatrix}$ is full, i.e. $rank Q_q = p$.

Proof: The result is the consequence of Lemma 6. The reasoning is similar to the one in a scalar fractional order case in Mozyrska et al. (2017). □

From the Rank Matrix Theorem and Proposition 8, it follows that $rank Q_q = p$ if and only if $q = p$. So, any state $x_0 \in R^p$ can be steered to a final state $x_f \in R^n$ in no more then p steps.

Definition 9: System (9) is *observable in q steps* if from the control sequence $U = \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u((q-1)h) \end{pmatrix}$ and the output sequence $Y = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y((q-1)h) \end{pmatrix}$ it is possible to determinate uniquely initial state x_0 of the given system.

Proposition 9: Let $\alpha = (\alpha_1, \dots, \alpha_p)$ with $\alpha_i \in (0,1]$, $i = 1, \dots, p$. Then, system (9) is observable in q steps if and only if the rank of observability matrix $W_q = \begin{pmatrix} \bar{C} \\ \bar{C}E_{(\alpha,\beta)}(\bar{A},1) \\ \vdots \\ \bar{C}E_{(\alpha,\beta)}(\bar{A},q-1) \end{pmatrix}$, where $\bar{C} := \begin{pmatrix} C & 0_{r\times p} & \dots & 0_{r\times p} \end{pmatrix}$ is full, i.e. $rank W_q = p$.

Proof: The result is the consequence of Lemma 6. The reasoning is similar to the one in a scalar fractional order case in Mozyrska et al. (2017). □

From the Rank Matrix Theorem and Proposition 10, it follows that $rank\, W_q = p$ if and only if $q = p$. So, based on the knowledge of control and output measurable sequences U and Y, respectively, the initial state $x_0 \in R^p$ can be uniquely determined in no more then p steps.

Controllable and observable triple (A, B, C) is called *canonical triple.*

4. REALISATION PROBLEM OF THE GIVEN IMPULSE

Consider system (9) with the initial state $x(t_0) = x_0$, vector-order $\alpha = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \in (0,1]$, $i = 1, \dots, p$ and given positive h. Observe that for any input $u: (hN)_{t_0} \to R^m$ and $t \geq t_0, t \in (hN)_{t_0}$, the following holds:

$$y(t) = \sum_{s=0}^{q=\frac{t-t_0}{h}} \Psi_\Lambda(t-sh)u(sh) = (\Psi_\Lambda * u)(t), \quad (12)$$

where $\Psi_\Lambda(t) = CE_{(\alpha,\beta)}(A, \frac{t-t_0}{h})B$, and $\beta = 1$ for the fractional h –difference of Caputo-type operator, $\beta = \alpha$ for the fractional h –differences of Riemman–Liouville- and Günwald–Letnikov-type operators. Function Ψ_Λ is called the *impulsive response* of system (9). Formula (12) defines the relation $S_{\Psi,q}$ between the input u and output y in q steps of the given control system, i.e.:

$$S_{\Psi,q}(u) = y. \quad (13)$$

Map $S_{\Psi,q}$ is called the *(q step) input–output map* of the considered system. Observe that between the impulsive response and the input–output map, there is a mutually inverse correspondence.

Suppose that $S_{\Psi,q}$ is an abstract q –steps input–output map acting on the input function u as

$$S_q(u) = \sum_{s=0}^{q=\frac{t-t_0}{h}} \Psi(t-sh)u(sh) = (\Psi * u)(t), \quad (14)$$

where map $\Psi: t \mapsto \Psi(t)$ is defined for all $t \in (hN)_{t_0}$. The problem is: *find a fractional vector-order state-space representation of map S_q in q steps.* In other words, for a chosen real positive h, we are looking for a linear fractional vector−order $\alpha = (\alpha_1, \dots, \alpha_p)$ control system (A, B, C), such that maps S_q and $S_{\Psi,q}$ coincide.

For the given abstract input–output map $S_q(u) = (\Psi * u)(t)$, define h –Markov parameters as

$$M_n^h := \Psi(nh + t_0) = \Psi(t),\ t \in (hN)_{t_0}. \quad (15)$$

Sequence $M^h = \{M_n^h : n \in N_0\}$ with elements M_n^h given by (15) will be called *h-Markov sequence* and its elements as *h-Markov parameters.*

Theorem 11: Let $h > 0$ and $\alpha = (\alpha_1, \dots, \alpha_p)$ with $\alpha_i \in (0,1]$, $i = 1, \dots, p$. Function $\Psi(t) = \Psi(nh + t_0) = \sum_{n=0}^{\infty} M_n^h$ is an impulsive characteristic of the fractional vector-order $\alpha = (\alpha_1, \dots, \alpha_p), \alpha_i \in (0,1]$, $i = 1, \dots, p$ system given by triple (A, B, C) if and only there are natural $\tilde{p}$ and real $a_0, a_1, \dots, a_{\tilde{p}-1}$, such that the following recursive relation holds:

$$M_{\tilde{p}+j}^h + a_{\tilde{p}-1}M_{\tilde{p}+j-1}^h + \dots + a_1 M_{1+j}^h + a_0 M_j^h = 0. \quad (16)$$

for $j = 0,1,2,3, \dots$.

Proof: Suppose that the input–output map (14) is a realisation of the fractional vector-order α system (A, B, C). Denote $\overline{\Psi}(n) := \Psi(nh + t_0)$ for any $n \in N$. So, there is α_i, $i = 1, \dots, p$, such that $\overline{\Psi}_i(p) = \bar{C}E_{(\alpha_i,\beta_i)}(\bar{A}, p)\bar{B}$ for some $\bar{A} \coloneqq \mathrm{diag}\{h^{-\alpha_i}A : i = 1, \dots, p\}$, $\bar{B} \coloneqq \begin{pmatrix} H^{-1}B & 0_{p\times m} & \dots & 0_{p\times m} \end{pmatrix}^{\mathrm{T}}$, $\bar{C} \coloneqq \begin{pmatrix} C & 0_{r\times p} & \dots & 0_{r\times p} \end{pmatrix}$ with $H = diag\{h^{-\alpha_i} : i = 1, \dots, p\}$. Thus, $M_p^h = \overline{\Psi}_i(p) = \bar{C}E_{(\alpha_i,\beta_i)}(\bar{A}, p)\bar{B}$. By Proposition 2 and formula (10) for any natural j, the following holds: $\bar{C}\left[E_{(\alpha,\beta)}(\bar{A},p)\right]^{\tilde{p}+j}\bar{B} + a_{p-1}\bar{C}\left[E_{(\alpha,\beta)}(\bar{A},p)\right]^{\tilde{p}+j-1}\bar{B} + \dots + a_0\bar{C}\left[E_{(\alpha,\beta)}(\bar{A},p)\right]^{j}\bar{B} = 0$.

Hence, (16) is fulfilled.

Now suppose that (16) holds for the given $\alpha = (\alpha_1, \dots, \alpha_p)$ with $\alpha_i \in (0,1]$, $i = 1, \dots, p$. Then, for

$$A = \begin{pmatrix} 0_{p\times p} & 0_{p\times p} & \dots & -a_0 h^{\alpha_1} I_p \\ h^{\alpha_2} I_p & 0_{p\times p} & \dots & -a_1 h^{\alpha_2} I_p \\ 0_{p\times p} & h^{\alpha_3} I_p & \dots & -a_1 h^{\alpha_2} I_p \\ \vdots & \vdots & \dots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \dots & -a_p h^{\alpha_p} I_p \end{pmatrix} \quad (17)$$

and

$$B = \begin{pmatrix} H^{-1} \\ 0_{p\times p} \\ \vdots \\ 0_{p\times p} \end{pmatrix} \quad (18)$$

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$$C = \begin{pmatrix} M_0^h & M_1^h & \dots & M_{p-1}^h \end{pmatrix}, \quad (19)$$

one obtains $M_p^h = CE_{(\alpha,\beta)}\left(A, p = \frac{t-t_0}{h}\right)B$. Hence, matrices A, B, C given by (17)–(19) define a realisation of the map $S_{\Psi,q}$. □

Example 12: Suppose that h is any real positive number, $t_0 = 0$ and $\alpha = (\alpha_1, \alpha_2)$, such that $\alpha_i \in (0,1]$, $i = 1,2$ and $\alpha_1 = \alpha_2$. Let $\Psi(t) = \Psi(nh) = \sum_{k=0}^{n=\frac{t}{h}} h^{-2\alpha_1}\binom{-k\alpha_1 - \beta_1}{n-k}$. Therefore, $M_n^h = \sum_{k=0}^{n} h^{-2\alpha_1}\binom{-k\alpha_1 - \beta_1}{n-k}$. Then, $M_{2+j}^h - M_{1+j}^h + \left(1 - h^{-\alpha_1} - \frac{\beta(\beta_1+1)}{2}h^{-\alpha_1}\right)M_j = 0$. The realisation for fractional h −differences of Riemman–Liouville- and Grünwald–Letnikov-type operators is given by the matrices $A = \begin{pmatrix} 0 & \frac{\alpha_1(\alpha_1+1)}{2}h^{2\alpha_1} \\ h^{\alpha_1} & -1 \end{pmatrix}$, $B = \begin{pmatrix} h^{-\alpha_1} \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} h^{-\alpha_1} & h^{-\alpha_1}(1 - 2\alpha_1) \end{pmatrix}$, and for fractional h −difference of Caputo-type operator, it is given by $A = \begin{pmatrix} 0 & h^{2\alpha_1} \\ h^{\alpha_1} & -1 \end{pmatrix}$, $B = \begin{pmatrix} h^{-\alpha_1} \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} h^{-\alpha_1} & -\alpha_1 h^{-\alpha_1} \end{pmatrix}$.

For a given h −Markov sequence M^h and positive integers s, ν, the block matrix

$$H_{s\nu}(M^h) = \begin{pmatrix} M_1^h & M_2^h & \dots & M_\nu^h \\ M_2^h & M_3^h & \dots & M_{\nu+1}^h \\ \vdots & \vdots & \ddots & \vdots \\ M_\nu^h & M_{\nu+1}^h & \dots & M_{s+\nu-1}^h \end{pmatrix} \quad (20)$$

is called the *Hankel matrix* associated with the sequence M^h.

Proposition 13: Let the triple (A, B, C) be described by (9a)–(9b). Then,

1. (A, B, C) realises M^h in q steps if and only if $W_q Q_q = H_{s\nu}(M^h)$ for all $s, \nu \in N$.
2. If (A, B, C) realises M^h in q steps and (A, B, C) is a canonical triple, then $rank\ H_{s\nu}(M^h) = p$ for $s, \nu \geq p$.

Proof: The result follows directly from propositions 8 and 10. □

In general, realisations are not unique. From a practical point of view, it is good to have such realisation for which the state-space has the possible minimal dimension, i.e. it is good to have a *minimal realisation*. This property is not easy for checking, but classically it is equivalent to the fact that triple (A, B, C) realising the h −Markov sequence should be canonical.

Theorem 14: If there is a realisation of h −Markov sequence M^h, then it is the canonical realisation.

Proof: The idea of the proof comes from Bartosiewicz and Pawluszewicz (2006). Suppose that system (9) is not controllable in a finite number of steps. So, there exist a natural number p_1 and a nonsingular matrix $P \in R^{p\times p}$, such that $P^{-1}AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A_{11} \in R^{p_1\times p_1}$, $A_{12} \in R^{p_1\times(p-p_1)}$ and $A_{22} \in R^{(p-p_1)\times(p-p_1)}$. Let $\bar{A}$ and $\bar{B}, \bar{C}$ be defined as in Lemma 6. So, by Proposition 2, it follows that $P^{-1}E_{(\alpha,\beta)}(\bar{A}, p)P = \begin{pmatrix} E_{(\alpha,\beta)}(\bar{A}_{11}, p) & E_{(\alpha,\beta)}(\bar{A}_{12}, p) \\ 0 & E_{(\alpha,\beta)}(\bar{A}_{22}, p) \end{pmatrix}$. Also, $P\bar{B} = \begin{pmatrix} \bar{B} \\ 0 \end{pmatrix}$ with $\bar{B}_1 \in R^{p_1\times m_1}$. So, $\bar{C}E_{(\alpha,\beta)}(\bar{A}, p)\bar{B} = \bar{C}PE_{(\alpha,\beta)}(\bar{A}, p)P^{-1}\bar{B} = \bar{C}_1 PE_{(\alpha,\beta)}(\bar{A}_{11}, p)P^{-1}\overline{B_1}$ for some matrix $\bar{C}$. So, system (9) is controllable in a finite number of steps.

The reasoning that system (9) is observable is the same. □

Corollary 15: A realisation of h −Markov sequence M^h is minimal if and only if it is canonical.

Proof: The implication " ⇒ " is the direct consequence of Proposition 13. The implication " ⇐ " follows from Theorem 14. □

5. CONCLUSIONS

The problem of realisation of the impulsive response function for fractional vector-order discrete time linear control systems was considered. It is shown that an abstract input–output map has a state-space realisation if and only if the h −Markov parameters satisfy the recurrence relation given by (16). This result extends the classical realisability criterion to fractional order systems. The description of state-space representation of input–output map is given in terms of fractional vector-order h −differences of Caputo-, Riemann–Liouville- and Grünwald–Letnikov-type operators. It is shown that the minimal fractional vector-order realisation exists if and only if triple (A, B, C) defining the state-space system is controllable and observable. Obtained results are illustrated by an academic example. Further work will focus on practical implementation of the obtained results in physical systems, including automatic control systems. Therefore, in a natural way also, more practical examples will appear.

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