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A CIRCULAR INCLUSION AND TWO RADIAL COAXIAL CRACKS WITH CONTACTING FACES IN A PIECEWISE HOMOGENEOUS ISOTROPIC PLATE UNDER BENDING

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Abstract: The bending problem of an infinite, piecewise homogeneous, isotropic plate with circular interfacial zone and two coaxial radial cracks is solved on the assumption of crack closure along a line on the plate surface. Using the theory of functions of a complex variable, complex potentials and a superposition of plane problem of the elasticity theory and plate bending problem, the solution is obtained in the form of a system of singular integral equations, which is numerically solved after reducing to a system of linear algebraic equations by the mechanical quadrature method. Numerical results are presented for the forces and moments intensity factors, contact forces between crack faces and critical load for various geometrical and mechanical task parameters.

Keywords: Bending, plate, interfacial zone, radial cracks, contact force, complex potentials, moment intensity factors, limit load

1. INTRODUCTION

Plate-shaped structural items are widely used in engineering. They may contain technological finite inclusions. There is also the possibility of cracking during operation. Cracks often greatly reduce plate's performance characteristics and may cause the structural item to destroy. In the presence of bending deformations, crack faces contact each other. It leads to significant redistribution of the stress-strain state near the crack tip (Shatsky, 1988; Kwon, 1989; Young and Sun, 1992; Dempsey et al., 1998; Opanasovych et al., 2012; Sulym et al., 2018) compared to neglecting the effect of crack closure.

Stress-strain state of biomaterial cracked plates and cracked plates with holes and inclusions under tension or/and bending is investigated by a variety of approaches and models (Wang and Nasebe, 2000; Hsieh and Hwu, 2002; Nielsen et al., 2012; Bäcker et al., 2015; Maksymovych and Illiushyn, 2017; Shao-Tzu and Li, 2017; Liu et al., 2018; Nguyen and Hwu, 2018; Sulym et al., 2018; Kuz' et al., 2019; Shiah et al., 2019 etc.).

Bending of a piecewise homogeneous, isotropic plate with a straight interfacial zone and a straight crack with contacting faces is investigated in Opanasovych and Slobodyan (2007).

The aim of this research is to investigate biaxial bending of a piecewise homogeneous isotropic plate with circular interfacial zone and two radial coaxial cracks on the assumption of crack closure along a line on one of the plate surfaces. Using methods of theory of functions of a complex variable together with complex potentials of classical plate bending theory and plane problem of elasticity theory, the solution of this problem is reduced to simultaneous singular integral equations, which are numerically solved. The forces and moments intensity factors, the contact forces between faces of cracks and the limiting plate load are analysed. Their graphical dependencies on various task parameters are plotted.

2. PROBLEM STATEMENT

Consider an infinite, piecewise homogeneous, isotropic plate with circular rigid inclusion and two coaxial radial cracks, whose faces are free from external loading. Let $2h$ is the plate thickness, R is the radius of the inclusion, and $2l_k$ is the length of the k^{th} crack ($k = 1, 2$). The plate is under the action of uniformly distributed bending moments at infinity. Suppose the crack faces smoothly contact alone a line on the upper surface of the plate.

The origin of the chosen Cartesian coordinate system $Oxy\tilde{z}$ is in the center of the circular rigid inclusion, the xy-plane coincides with the middle plane of the plate and the cracks are oriented along the x-axis. In the xy-plane, we introduce the polar coordinates (r, θ) with pole O and polar axis Ox. The x-coordinates of crack centres are $x_1 = R + d_1 > R + l_1$ and $x_2 = -R - d_2 < -R - l_2$, where d_k is a distance from the centre of the k^{th} crack to the interfacial line. The x-coordinates of cracktips are a_i and $b_i (i = 1, 2)$. In the middle plane $S^+(S_1)$and $S^-(S_2)$ refer to the areas inside and outside the inclusion, respectively, L_1 denotes the union of straight line segments of two cracks, and L – the interfacial contour. M_x^∞ and M_y^∞ stand for distributed bending moments at infinity (Fig. 1).

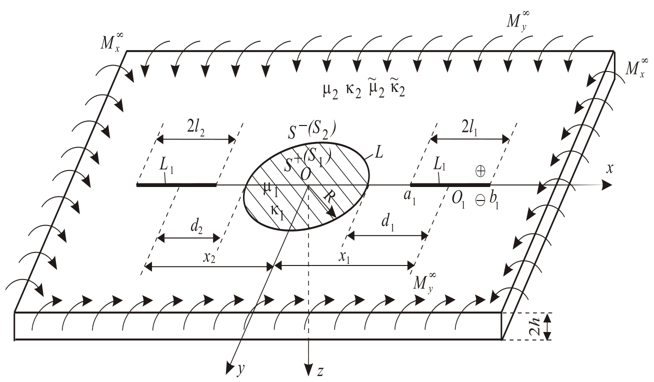

Fig. 1. Plate geometries and load scheme

Due to crack closure, the solution is a superposition of the solutions of two problems (Shatsky, 1988): the classical bending problem and the plane problem of elasticity theory under the following boundary conditions:

$$\sigma_{yy}^{\pm}=-\frac{N}{2h},\sigma_{xy}^{\pm}=P_y^{\pm}=0,\ M_y^{\pm}=M_y=hN,x\in L_1, \quad (1)$$

$$\partial_x\left[u_y\right]+h\left[\partial_{xy}^2 w_2\right]=0,x\in L_1, \quad (2)$$

$$P_{r1}=P_{r2},\ M_{r1}=M_{r2},\ (r,\theta)\in L, \quad (3)$$

$$u_{r1}=u_{r2},\ u_{\theta 1}=u_{\theta 2},(r,\theta)\in L, \quad (4)$$

$$w_1=w_2,\partial_r w_1=\partial_r w_2,\ (r,\theta)\in L, \quad (5)$$

where: N – contact force between crack faces, σ_{xy} and σ_{yy} – stress tensor components,$u_{\theta j}$ and u_y – displacement vector components of plane problem (here and further $j=1,2$), w_j – deflection of the plate, M_{rj} and M_y– bending moments, P_y and P_{rj}– generalized Kirchhoff shear forces, $[f]=f^+-f^-$ (superscripts '+' i '–' stand for limits of function f as a point of the middle plane approaches the cracks, $y\to\pm 0$), $\partial_a=\partial/\partial a$.

3. SOLUTION OF PLATE BENDING PROBLEM

We introduce complex potentials (Prusov, 1975) for areas S_j and set them as follows:

$$\Phi_{3j}(z)=\Phi_3^{(j)}(z)+\widetilde{\Phi}_1(z)+\tilde{\Gamma},$$

$$\Psi_{3j}(z)=\Psi_3^{(j)}(z)+\widetilde{\Psi}_1(z)+\tilde{\Gamma}',$$

where: $z=x+iy, i=\sqrt{-1}, \Phi_3^{(j)}(z)$ and $\Psi_{3j}(z)$ – holomorphic in S_j functions, $\widetilde{\Phi}_1(z)$ and $\widetilde{\Psi}_1(z)$ – vanished at infinity functions, which are holomorphic outside the cracks, $\tilde{\Gamma}=-\frac{M_y^\infty+M_x^\infty}{4D_2(1+\nu_2)}, \tilde{\Gamma}'=\frac{M_y^\infty-M_x^\infty}{2D_2(1-\nu_2)}, D_j=\frac{2Q_j}{3\left(1-\nu_j^2\right)},\ Q_j=E_j h^3,\ E_j$ – elastic modulus, ν_j – Poisson's ratio.

Using the functions (Prusov, 1975) $\widetilde{\Omega}_1(z)=-\overline{\widetilde{\Phi}}_1(z)-z\overline{\widetilde{\Phi}}_1{}'(z)-\overline{\widetilde{\Psi}}_1(z)$ and $\Phi_3^{(j)}(z)=-\overline{\Phi}_3^{(j)}\left(\frac{R^2}{z}\right)+\frac{R^2}{z}\overline{\Phi}_3^{(j)\prime}\left(\frac{R^2}{z}\right)+\frac{R^2}{z^2}\overline{\Psi}_3^{(j)}\left(\frac{R^2}{z}\right)$, in which $z\in S_{3-j}$, we can express the basic formulas of the classical plate bending theory in the form:

$$2\tilde{\Gamma}-\frac{\bar{z}}{z}\tilde{\Gamma}'+\Phi_3^{(j)}(z)-f_3^{(j)}(z)+\widetilde{\Phi}_1(z)+\tilde{f}_1(z)=\tilde{g}_j, \quad (6)$$

$$\left(\tilde{\kappa}_j-1\right)\tilde{\Gamma}+\frac{\bar{z}}{z}\tilde{\Gamma}'+\tilde{\kappa}_j\Phi_3^{(j)}(z)+f_3^{(j)}(z)+\tilde{\kappa}_j\widetilde{\Phi}_1(z)-\tilde{f}_1(z)=\tilde{f}_j, \quad (7)$$

$$\left(\tilde{\kappa}_2-1\right)\tilde{\Gamma}-\tilde{\Gamma}'+\tilde{\kappa}_2\widetilde{\Phi}_1(z)+\tilde{f}_2(z)+\tilde{\kappa}_2\Phi_3^{(2)}(z)-g_3^{(2)}(z)=f_2, \quad (8)$$

$$2\tilde{\Gamma}+\tilde{\Gamma}'+\widetilde{\Phi}_1(z)-\tilde{f}_2(z)+\Phi_3^{(j)}(z)-g_3^{(2)}(z)=\partial_x g, \quad (9)$$

where:

$$\tilde{f}_1(z)=\left(1+\frac{\bar{z}}{z}\right)\widetilde{\Phi}_1(z)+\frac{\bar{z}}{z}\tilde{f}_2(z),$$

$$\tilde{f}_2(z)=\widetilde{\Omega}_1(\bar{z})-(z-\bar{z})\overline{\widetilde{\Phi}_1'(z)},$$

$$f_3^{(j)}(z)=\frac{R^2}{r^2}\Phi_3^{(j)}\left(\frac{R^2}{\bar{z}}\right)-\left(1-\frac{R^2}{r^2}\right)\left\{\overline{\Phi_3^{(j)}(z)}-\bar{z}\overline{\Phi_3^{(j)\prime}(z)}\right\},$$

$$g_3^{(2)}(z)=\left(1+\frac{R^2}{\bar{z}^2}\right)\overline{\Phi_3^{(2)}(z)}+z\overline{\Phi_3^{(2)\prime}(z)}-\frac{R^2}{\bar{z}^2}\left\{\Phi_3^{(2)}\left(\frac{R^2}{\bar{z}}\right)-\bar{z}\overline{\Phi_3^{(2)\prime}(z)}\right\}, z=re^{i\theta}, \tilde{\kappa}_j=\left(3+\nu_j\right)/\left(1-\nu_j\right),$$

$$g=\partial_x w_2+i\partial_y w_2,$$

$$\tilde{g}_j=-\frac{i}{z}\partial_\theta\left(\left(\partial_r w_j+\frac{i}{r}\partial_\theta w_j\right)e^{i\theta}\right),$$

$$\tilde{f}_j=2\tilde{\mu}_j\left\{-M_r-ic_j'-iH_{r\theta}-i\int_0^s N_r(\tau)d\tau\right\},$$

$$f_2=-2\tilde{\mu}_2\left\{M_y+i\tilde{c}'+iH_{xy}+i\int_{-l_j}^{\tau}N_y(\tau)d\tau\right\},$$

$\tilde{\mu}_j=1/\left(2D_j\left(1-\nu_j\right)\right)$, c_j' and $\tilde{c}'$– real constants.

If the expansions of function $\Phi_3^{(1)}(z)$ and its analytic continuation in a series $\Phi_3^{(j)}(z)=\tilde{A}_0'+\tilde{A}_1'z+\ldots(z\to 0)$ and $\Phi_3^{(1)}(z)=\tilde{B}_0'+\tilde{B}_1'z^{-1}+\ldots(z\to\infty)$ are valid, the conditions (Prusov, 1975) $\tilde{B}_1'=0$ and $\tilde{B}_0'=-\overline{\tilde{A}_0'}$ are fulfilled too.

On account of boundary value problem (1)–(2) and formula (8), we obtain a linear conjugation problem:

$$\left(\tilde{\kappa}_2\widetilde{\Phi}_1(t)-\widetilde{\Omega}_1(t)\right)^+-\left(\tilde{\kappa}_2\widetilde{\Phi}_1(t)-\widetilde{\Omega}_1(t)\right)^-=0, t\in L_1$$

whose solution is:

$$\widetilde{\Omega}_1(z)=\tilde{\kappa}_2\widetilde{\Phi}_1(z). \quad (10)$$

On the basis of (9), taking into account representation (10) and boundary conditions (1)–(2), we form the following linear conjugation problem:

$$\widetilde{\Phi}_1^+(t)-\widetilde{\Phi}_1^-(t)=Q_1(t), t\in L_1.$$

The solution of this problem is:

$$\widetilde{\Phi}_1(z)=\frac{1}{2\pi i}\int_{L_1}\frac{Q_1(t)}{t-z}dt,$$

where $Q_1(t)=\partial_x\left[\partial_x w_2+i\partial_y w_2\right]/\left(1+\tilde{\kappa}_2\right)$.

From the boundary conditions (5) and formula (6). we obtain one more linear conjugation problem:

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$$\left(\Phi_3^{(1)}(t)+\Phi_3^{(2)}(t)\right)^+-\left(\Phi_3^{(1)}(t)+\Phi_3^{(2)}(t)\right)^-=0, t\in L$$

with the solution:

$$\Phi_3^{(1)}(z)+\Phi_3^{(2)}(z)=-\overline{\tilde{A}_0'}. \qquad (11)$$

Introducing a function:

$$\tilde{\Phi}(z)=\begin{cases} ic-\left(\underline{AA}_3+\underline{AA}_4\right)\tilde{\Gamma}+\tilde{F}_1(z)+F_3^{(1)}(z), \\ z\in S^+, \\ -\underline{AA}_4\frac{R^2}{z^2}\tilde{\Gamma}'+\tilde{F}_2(z)+F_3^{(2)}(z), z\in S^-, \end{cases} \qquad (12)$$

where: $\tilde{F}_1(z)=-\underline{AA}_3\tilde{\Phi}_1(z)$, $F_3^{(j)}(z)=\tilde{\mu}_{3-j}\tilde{\kappa}_j\Phi_3^{(j)}(z)-\tilde{\mu}_j\Phi_3^{(3-j)}(z)$, $\tilde{F}_2(z)=\underline{AA}_4\left\{\left(1+\frac{R^2}{z^2}\right)\overline{\tilde{\Phi}}_1\left(\frac{R^2}{z}\right)+\frac{R^2}{z}\left\{\tilde{\kappa}_2\tilde{\Phi}_1\left(\frac{R^2}{z}\right)-\left(z-\frac{R^2}{z}\right)\overline{\tilde{\Phi}}_1'\left(\frac{R^2}{z}\right)\right\}\right\}$, $c=2\tilde{\mu}_1\tilde{\mu}_2(c_1'-c_2')$, $\tilde{g}=-\tilde{A}_1/\tilde{A}_2$, $\tilde{A}_j=\tilde{\mu}_j+\tilde{\mu}_{3-j}\tilde{\kappa}_j$, $\underline{AA}_3=\tilde{\mu}_1\tilde{\kappa}_2-\tilde{\mu}_2\tilde{\kappa}_1$, $\underline{AA}_4=\tilde{\mu}_2-\tilde{\mu}_1$, with respect to the boundary conditions (3) and formula (7), we make sure it is a solution of the linear conjugation problem $\tilde{\Phi}^+(t)-\tilde{\Phi}^-(t)=0$ $(t\in L)$, which can be written as

$$\tilde{\Phi}(z)=\tilde{B}+\tilde{\mu}_2\overline{\tilde{A}_0'}, \qquad (13)$$

where: $\tilde{B}=i\underline{AA}_4\overline{B_1}$, $B_1=\frac{1}{2\pi}\int_{L_1}t^{-1}Q_1(t)dt$.

On the basis of (11) and (12) with respect to (13), we obtain:

$$\Phi_3^{(1)}(z)=-\Phi_3^{(2)}(z)-\overline{\tilde{A}_0'}, z\in S_j,$$

$$\Phi_3^{(2)}(z)=\begin{cases} \frac{1}{\tilde{A}_1}\{\tilde{F}_1(z)+ic-\tilde{B}\}-\left(\frac{\tilde{A}_3}{\tilde{g}}+\tilde{A}_4\right)\tilde{\Gamma}- \\ \tilde{A}_5\overline{\tilde{A}_0'},\ z\in S^+, \\ \frac{1}{\tilde{A}_2}\{\tilde{B}-F_2(z)\}-\tilde{g}\tilde{A}_4\frac{R^2}{z^2}\tilde{\Gamma}',\ z\in S^-, \end{cases} \qquad (14)$$

where: $\tilde{A}_4=\underline{AA}_4/\tilde{A}_1$, $\tilde{A}_3=\underline{AA}_3/\tilde{A}_2$, $\tilde{A}_5=\tilde{\mu}_2(1+\tilde{\kappa}_1)/\tilde{A}_1$.

Since $\Phi_3^{(1)}(0)=\tilde{A}_0'$, in view of (14), we can write:

$$\mathrm{Re}\tilde{A}_0'=\frac{\tilde{A}_{12}}{1-\tilde{A}_4}\left(\tilde{\Gamma}+\mathrm{Im}\,B_1\right), \frac{c}{\tilde{A}_1}+\tilde{A}_5\mathrm{Im}\,\tilde{A}_0'=\tilde{a}\mathrm{Re}B_1,$$

where: $\tilde{A}_{12}=\tilde{A}_4-\tilde{A}_3/\tilde{g}$, $\tilde{a}=\tilde{A}_3/\tilde{g}+\tilde{A}_4$.

From the boundary conditions (1)–(2) and formula (8), we finally obtain the following integral equations:

$$\sum_{k=1}^{2}\int_{-1}^{1}\{Y_{k1}[K_{mk}(\eta,\xi)+L_{mk}(\eta,\xi)]\}d\eta=\tilde{c}_m', \qquad (15)$$

$$\sum_{k=1}^{2}\int_{-1}^{1}\{Y_{k2}N_{mk}(\eta,\xi)\}d\eta=\tilde{m}hN_m(\xi)/M_y^\infty+P_m(\xi), \qquad (16)$$

where:

$$Y_k(t)=Q_1Q_2(l_kt+x\_k)/M_y^\infty=Y_{k1}(t)+Y_{k2}(t),$$

$$\tilde{m}=-1/\left(\tilde{D}_2(1-\nu_2)\right), \tilde{D}_2=2/\left(3\,(1-\nu_2^2)\right),$$

$$P_m(\xi)=-\tilde{m}+\frac{\tilde{g}\tilde{A}_4B}{X_m^2}\left(\tilde{\kappa}_2+1-\frac{3}{X_m^2}\right)-\frac{2A\tilde{A}_{12}}{X_m^2(1-\tilde{A}_4)},$$

$$A=-(\rho+1)/\left(4\tilde{D}_2(1+\nu_2)\right), B=-\tilde{m}(1-\rho)/2,$$

$$K_{mk}(\eta,\xi)=-\frac{1}{\pi}\left\{\tilde{\gamma}_2\tilde{\lambda}_k\tilde{K}_{mk}(\eta,\xi)+\frac{\tilde{\lambda}_k}{2}\left\{\frac{\tilde{g}\tilde{A}_4}{T_k}\left(1+\frac{1}{X_m^2}\right)+\frac{1}{2X_m^2}\left(\tilde{a}-\frac{\tilde{A}_5\tilde{A}_{12}}{\tilde{A}_4-1}\right)+Q_{km}\left[\tilde{g}\tilde{A}_4\left(\frac{\tilde{\gamma}_2^2}{X_m}-X_m-\frac{3}{X_m^3}\right)-\frac{\tilde{A}_3}{\tilde{g}X_m}\right]+\tilde{g}\tilde{A}_4Q_{km}^2\left(X_m+\frac{4}{X_m}-\frac{5}{X_m^3}\right)-\frac{2\tilde{g}\tilde{A}_4}{X_m}\left(X_m-\frac{1}{X_m}\right)^2Q_{km}^3\right\}\right\},$$

$$\tilde{K}_{mk}(\eta,\xi)=(T_k-X_m)^{-1}, \tilde{\gamma}_1=1+\xi_1, \tilde{\gamma}_2=-1-\xi_2,$$

$$T_k=\tilde{X}_k+\tilde{\lambda}_k\eta, X_m=\tilde{X}_m+\tilde{\lambda}_m\xi, \tilde{\lambda}_k=l_k/R, \xi_k=d_k/R,$$

$$\rho=M_x^\infty/M_y^\infty, N_{mk}(\eta,\xi)=L_{mk}(\eta,\xi)-K_{mk}(\eta,\xi),$$

$$L_{mk}(\eta,\xi)=-\frac{\tilde{\lambda}_k}{2\pi}\left\{\frac{1}{T_k}\left(\tilde{\kappa}_2\tilde{g}\tilde{A}_4-\frac{1}{X_m^2}\left(\tilde{A}_4-\frac{\tilde{a}}{2}-\frac{\tilde{A}_5\tilde{A}_{12}}{2(\tilde{A}_4-1)}\right)\right)+\tilde{\kappa}_2\tilde{g}\tilde{A}_4Q_{km}\left[\frac{3}{X_m^3}-X_m-\frac{2}{X_m}+Q_{km}\left(X_m-\frac{2}{X_m}+\frac{1}{X_m^3}\right)\right]\right\},$$

$Y_{k1}(t), Y_{k2}(t)$– real functions, $Q_{km}=1/(T_kX_m-1)$.

Equations (15) and (16) must be solved under the additional conditions:

$$\int_{-1}^{1}Y_k(\eta)d\eta=\int_{-1}^{1}\eta Y_{k1}(\eta)d\eta=0\,, k=1,2, \qquad (17)$$

which assume that rotational displacements and deflection of the plate have to be single-valued when bypassing the contours of cracks.

Note that if the crack closure is neglected, the system of singular integral equations (15)–(17) takes $N_m(\xi)=0$.

4. SOLUTION OF PLANE PROBLEM

We introduce Kolosov–Muskhelishvili complex potentials (Muskhelishvili, 1966) for areas S_j and represent them in the form:

$$\Phi_{Pj}(z)=\Phi_P^{(j)}(z)+\Phi_1(z), \Psi_{Pj}(z)=\Psi_P^{(j)}(z)+\Psi_1(z),$$

where: $\Phi_1(z)$, $\Psi_1(z)$ – vanished at infinity functions, which are holomorphic outside the cracks;$\Phi_P^{(j)}(z)$, $\Psi_P^{(j)}(z)$ – holomorphic functions in S_j. Moreover, at large $|z|\Phi_P^{(2)}(z)=O(1/z^2)$ and $\Psi_P^{(2)}(z)=O(1/z^2)$.

Similar as the previous chapter, we also introduce the following functions (Prusov, 1962):

$$\Phi_P^{(j)}(z)=-\overline{\Phi}_P^{(j)}\left(\frac{R^2}{z}\right)+\frac{R^2}{z}\overline{\Phi}_P^{(j)\prime}\left(\frac{R^2}{z}\right)+\frac{R^2}{z^2}\overline{\Psi}_P^{(j)}\left(\frac{R^2}{z}\right)$$

$$\tilde{\Omega}_1(z)=-\overline{\tilde{\Phi}}_1(z)-z\overline{\tilde{\Phi}}_1'(z)-\overline{\tilde{\Psi}}_1(z), z\in S_{3-j}.$$

Then a stress-strain state of the plate is given by the equations:

$$\sigma_{rr}+i\sigma_{r\theta}=\Phi_P^{(j)}(z)-f_P^{(j)}(z)+f_1(z), \qquad (18)$$

$$2\mu_j\partial_\theta\left(u_x+iv_y\right)=iz\left[\kappa_j\Phi_{Pj}(z)+f_P^{(j)}(z)-f_1(z)\right], \qquad (19)$$

$$\sigma_{yy}-i\sigma_{xy}=\Phi_{Pj}(z)+f_2(z)+g_P^{(2)}(z), \qquad (20)$$

$$2\mu_2\partial_x\left(u_x+iv_y\right)=\kappa_2\Phi_{Pj}(z)-f_2(z)-g_P^{(2)}(z), \qquad (21)$$

where: $\kappa_j=(3-\nu_j)/(1+\nu_j)$, $\mu_j=E_j/\left(2(1+\nu_j)\right)$–

shear modulus, $f_1(z) = (1+\bar{z}/z)\Phi_1(z) - \bar{z}/z\, f_2(z)$, $f_2(z) = \Omega_1(\bar{z}) + (z-\bar{z})\overline{\Phi_1'(z)}$, functions $f_P^{(j)}(z)$ and $g_P^{(2)}(z)$ can be obtained from expressions for $f_3^{(j)}(z)$ and $g_3^{(2)}(z)$ from (7), (9) by replacing index '3' by 'P'.

Formulas (6)–(9) in the bending problem and corresponding dependencies (18)–(21) in plane problem have the same structure. The boundary conditions (1)–(5) are also similar for both problems. Hence, by using the approach from the previous chapter, we find:

$$\Phi_1(z) = \Omega_1(z) = \frac{1}{2\pi i}\int_{L_1}\frac{g'(t)}{t-z}dt,$$

$$\Phi_P^{(1)}(z) = -\Phi_P^{(2)}(z) - \overline{A_0'},\, z \in S_j,$$

$$\Phi_P^{(2)}(z) = \begin{cases}(F_1(z)-B)/A_1 - A_5\overline{A_0'}, z\in S^+,\\ \left(B-F_2(z)\right)/A_2\,, z\in S^-,\end{cases}$$

where: $g'(x) = \frac{2\mu_2}{1+\kappa_2}\partial_x[u_x + iu_y]$, $A_5 = \frac{\mu_2}{A_1}(1+\kappa_1)$, $A_0' = \frac{B_5}{2\pi i}\int_{L_1}\frac{1}{t}\left[\left(A_4^2 - \frac{A_3}{g}\right)g'(t) + A_4\left(\frac{A_3}{g}-1\right)\overline{g'(t)}\right]dt$, $B_5 = (1-A_4^2)^{-1}$, expressions for B, $F_1(z)$, $F_2(z)$, $\widetilde{AA}_3$, $\widetilde{AA}_4$, g, $A_n (n=1,4)$ are obtained from the corresponding expressions for $\tilde{B}$, $\tilde{F}_1(z)$, $\tilde{F}_2(z)$, $\underline{AA}_3$, $\underline{AA}_4$, $\tilde{g}$, $\tilde{A}_n (n=1,4)$ by the substitution $Q_1(t) \to g'(t)$, $\tilde{\Phi}_1(z) \to \Phi_1(z)$, $\tilde{\mu}_k \to \mu_k$, $\tilde{\kappa}_k \to \kappa_k$, $\tilde{A}_k \to A_k\ (k=1,2)$.

In view of the boundary conditions (1)–(2), an unknown derivative of displacement jump across the crack faces $g'(x)$ is obtained by solving the integral equations:

$$\sum_{k=1}^{2}\int_{-1}^{1}G_{k2}(\eta)[R_{mk}(\eta,\varepsilon) - S_{mk}(\eta,\varepsilon)]d\eta = 0, \tag{22}$$

$$\sum_{k=1}^{2}\int_{-1}^{1}G_{k1}(\eta)M_{mk}(\eta,\varepsilon)d\eta = -\pi h N_m(\varepsilon)/\left(2M_y^{\infty}\right), \tag{23}$$

at $|\varepsilon|<1$, $m=1,2$ and satisfying that displacements have to be single-valued when bypassing the contour of each crack:

$$\int_{-1}^{1}G_k(\eta)d\eta = 0,\ k=1,2. \tag{24}$$

Formulas (20)–(22) have the following notations:

$$R_{mk}(\eta,\varepsilon) = \lambda_k\left\{\tilde{K}_{mk}(\eta,\varepsilon) - \frac{A_4 g Q_{km}}{2\tilde{X}_m}\left\{\frac{1}{X_m} - \left(\frac{1}{X_m^2} + 1\right)\left[X_m + \frac{1}{X_m} - \tilde{X}_m Q_{km}\right] + 2\tilde{X}_m\left[\frac{2Q_{km}}{X_m^2} + \frac{1}{X_m^2} - \frac{\tilde{X}_m Q_{km}^2}{X_m}\right]\right\}\right\},$$

$M_{mk}(\eta,\varepsilon) = R_{mk}(\eta,\varepsilon) + S_{mk}(\eta,\varepsilon)$, $\tilde{X}_m = X_m - 1/X_m$, $S_{mk}(\eta,\varepsilon) = -\frac{\tilde{\lambda}_k}{2}\left\{\frac{1}{T_k}\left(\frac{B_9}{X_m^2} + gA_4\right) - gA_4Q_{km}\left[X_m - \frac{1}{X_m^3} - \tilde{X}_m Q_{km} + \frac{Q_{km}+2}{X_m^2}\right]\right\}$, $G_k(\eta) = \frac{h^2}{M_y^{\infty}}g'(l\eta) = G_{k1}(\eta) + iG_{k2}(\eta)$, $B_8 = A_5B_5\left(\frac{A_3}{g}-1\right)$, $B_9 = A_4 + A_5B_5\left(A_4^2 - \frac{A_3}{g}\right)$.

5. SUPERPOSITION OF SOLUTIONS

By substituting $N_{mk}(\varepsilon)$, which is obtained from (23) into (16), we get:

$$\sum_{k=1}^{2}\int_{-1}^{1}\{Y_{k2}N_{mk}(\eta,\varepsilon) + \beta_1 G_{k1}(\eta)M_{mk}(\eta,\varepsilon)\}d\eta = P_m(\varepsilon), |\varepsilon|<1, m=1,2, \tag{25}$$

where $\beta_1 = 2\tilde{m}/\pi$.

Satisfying the boundary condition (2) leads to:

$$Y_{k2}(\eta) = \beta G_{k1}(\eta), \tag{26}$$

where: $\beta = -(1+\kappa_2)(1+\nu_2)/(1+\tilde{\kappa}_2)$.

Based on the analysis of system of equations (15), (17), (22)–(24), (25) and (26) we conclude that $c_k' = 0$, $G_{k2}(\eta) = Y_{k1}(\eta) = 0\ (k=1,2)$, that is, the solution of the problem is reduced to a system of singular integral equations, which consists of the following equation:

$$\sum_{k=1}^{2}\int_{-1}^{1}\{\beta N_{mk}(\eta,\varepsilon) + \beta_1 M_{mk}(\eta,\varepsilon)\}G_{k1}(\eta)d\eta = P_m(\varepsilon), |\varepsilon|<1, m=1,2, \tag{27}$$

and equation (24).

Note that at $E_1 = 0$, this system turns into the system of integral equations from the research by Opanasovych and Slobodyan (2007).

6. NUMERICAL ANALYSIS

By using the mechanical quadrature method (Panasyuk et al., 1976), the system of singular integral equations (27), (24) is reduced to the following system of linear algebraic equations:

$$\frac{\pi}{M}\sum_{k=1}^{2}\sum_{m=1}^{M}Y_{km}[\beta N_{mk}(\eta_m,\varepsilon_r) + \beta_1 M_{mk}(\eta_m,\varepsilon_r)]d\eta = P_m(\varepsilon_r), m=1,2, r=\overline{1,M-1},$$

$$\sum_{m=1}^{M}Y_{km}(\eta) = 0,\ k=1,2,$$

where $Y_{km} = \sqrt{1-\mu^2}G_{k1}(\eta_m)$, $\eta_m = \cos\frac{(2m-1)\pi}{2M}$, $\varepsilon_r = \cos\frac{\pi r}{M}$.

The crack-tip stress distribution is given in research by Panasyuk et al. (1976). Formulas for the reduced moments intensity factors are:

$$K_M^{*\pm} = \frac{K_M^{\pm}}{M_y^{\infty}\sqrt{l}} = \mp\frac{2}{\beta_2(1-\nu_2)M}\sum_{m=1}^{M}(-1)^{m+1+\frac{(M-1)}{2}(1\mp 1)}Y_{km}\cot^{\mp 1}\frac{(2m-1)\pi}{4M},$$

Where: $K_M^{\pm}$ are the bending moment intensity factors (twisting moment intensity factors are equal to 0); $\beta_2 = 3(1+\nu_2)/(3+\nu_2)$, '+' and '–' correspond to tips b_i and $a_i (i=1,2)$, respectively.

Note that reduced forces intensity factors $K_N^{*\pm} = \frac{hK_N^{\pm}}{M_y^{\infty}\sqrt{l}}$ are related to $K_M^{*\pm}$ as $K_N^{*\pm} = \beta_2 K_M^{*\pm}$, where $K_N^{\pm}$ are the forces intensity factors.

The critical value of the moment at which the plate collapses is calculated by the formula (Osadchuk, 1985):

$$\tilde{M}^{\pm} = \frac{M_y^{\infty}}{2h}\sqrt{\frac{\pi l}{2\gamma_* E_2}} = \left(K_M^{*\pm}\sqrt{\beta_2^2 + \beta_2}\right)^{-1},$$

where: γ_* is the density of an active surface energy of the plate material.

Numerical analysis is carried out at $\nu_1 = \nu_2 = 0.3$ and $l_1 = l_2 = l$. The values of $\tilde{n} = E_1/E_2$ are 0.1, 0.5, 1, 2, 10, 0.001, 1000 for lines labelled by 1, 2, 3, 4, 5, 6, and 7, respectively. In Figures 3 and 4, dashed lines correspond to the case when crack closure is neglected.

Fig. 2 illustrates the graphical dependence of the reduced contact force $N^* = hN/M_y^{\infty}$ between crack faces on the dimen-

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sionless coordinate $\xi = x_1/l$ at $d_1 = d_2 = d$, $\varepsilon = d/R = 1$, $\lambda = l/R = 0.8$ and $M_x^\infty/M_y^\infty = 1$.

Graphical dependencies of the reduced moment intensity factor K_M^* on $\varepsilon = d/R$ for tips a and b at $d_1 = d_2 = d$, $\lambda = l/R = 0.8$ and $M_x^\infty/M_y^\infty = 0.5$ are shown in Fig. 3.

Fig. 4 presents the graphical dependence of the reduced critical moment $\widetilde{M}$ on the relative distance from the second crack to the interface $\varepsilon_2 = d_2/R$ at $\lambda = l/R = 0.7$, $\varepsilon_1 = d_1/R = 1$ and $M_x^\infty/M_y^\infty = 1$.

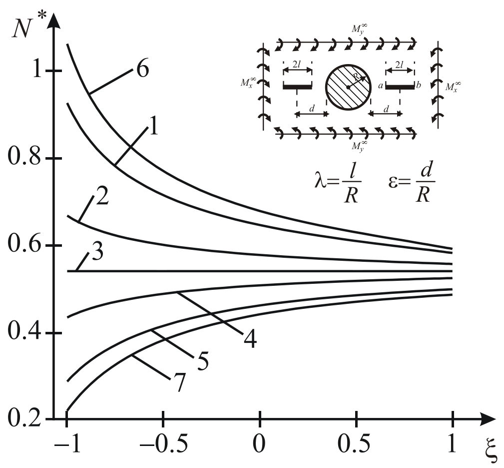

Fig. 2. Dependence of the reduced contact force on the distance between interface and cracks

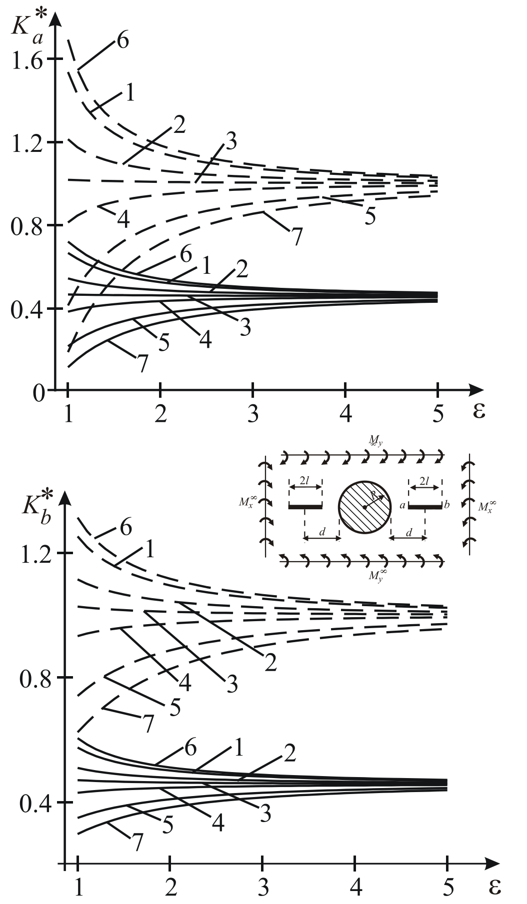

Fig. 3. Dependences of the reduced moment intensity factor on $\varepsilon = d/R$ in tip a (K_a^*) and b (K_b^*)

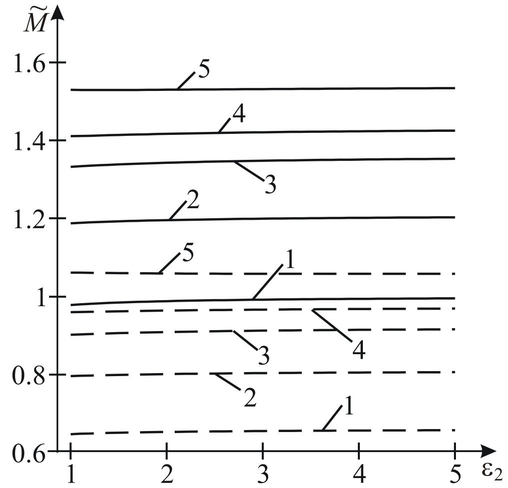

Fig. 4. Dependence of the reduced critical moment on the relative distance between second crack and interfacial line

7. CONCLUSIONS

The obtained dependencies show that if the inclusion is more rigid than the plate, the values of reduced contact force, intensity factors and critical moment are smaller than the corresponding values in case of a homogeneous plate. The situation is reversed for the pliable (in comparison with the plate) inclusion. The highest values are reached for the hole, the minimal ones – for the rigid plate.

Taking into account the contact of the crack faces leads to a decrease in the coefficients of the moment intensity and an increase in the ultimate load compared to the case when the contact of the crack faces is not taken into account. Crack closure consideration leads to decreasing of the moment intensity factors and to increasing of limit load.

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